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Characterizations of Certain Classes of Semicontinuous Multifunctions by Continuous Approximations

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1. INTRODUCTION

In 1967 Hukuhara [6] established a fundamental theorem on the approximation of an Hausdorff upper semicontinuous multifunction $F: X \to \mathscr{U}_k(\mathbb{R}^d)$ by a monotone decreasing sequence of continuous multifunctions $G_n: X \to \mathscr{U}_k(\mathbb{R}^d)$ converging pointwise to F and satisfying $G_n(x) \supset F(x)$, for each $x \in X$. Here, X is a nonempty closed subset of the real l-dimensional Euclidean space $\mathbb{R}^l$ and $\mathscr{U}_k(\mathbb{R}^d)$ denotes the space of all nonempty convex compact subsets of $\mathbb{R}^d$ endowed with Hausdorff distance. As shown in [6], the theorem of Hukuhara has useful and interesting applications: for instance, it is the main tool to generalize degree theory for Hausdorff upper semicontinuous multifunctions with nonempty convex compact values; in addition, it makes it possible to give alternative proofs of the classical fixed point theorems of Kakutani and Fan [7, 3].

More recently, new applications of Hukuhara's theorem have been discovered. In fact, using Hukuhara's theorem, Lasry and Robert [8, p. 123] have proved (among other things) that the solution sets of certain types of differential equations with delay are acyclic. Moreover, Himmelberg and Van Vleck [5] have applied Hukuhara's theorem in order to prove that the solution sets of certain types of multivalued differential equations are R_δ-sets. Problems of this kind have been investigated (by other methods) by Szufla [11, 12] in relation to differential and integral equations in Banach spaces. In 1981, Haddad [4] (see also Lasry and Robert [8, p. 124]) extended Hukuhara's theorem to the class of Hausdorff upper semicontinuous compact multifunctions from a metric space X to the space of nonempty convex compact subsets of a real normed space endowed with Hausdorff distance. Haddad also presents some applications of his result which are in part in the spirit of Hukuhara [6, Section 17] and in part devoted to the

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study of the solution sets of multivalued functional differential equations in $\mathbb{R}^d$.

In this paper, following Hukuhara's ideas, we obtain some generalizations of Hukuhara's approximation theorem. We shall consider multifunctions $F: X \to \mathscr{Z}$, where X is a metric space and $\mathscr{Z}$ stands for any of the spaces $\mathscr{U}_k(Z)$, $\mathscr{U}_c(Z)$, $\mathscr{U}_0(Z)$ (each endowed with Hausdorff distance h) consisting of all nonempty subsets of a real normed space Z which are, respectively, convex compact, convex closed and bounded, and convex closed and bounded with nonempty interior. We shall prove that (Theorem 4.5) for $\mathscr{Z} = \mathscr{U}_c(Z)$ or $\mathscr{Z} = \mathscr{U}_0(Z)$ the Hausdorff upper semicontinuous multifunctions $F: X \to \mathscr{Z}$ are characterized by the following property: $(*)$ there is a sequence $\{G_n\}$ of continuous multifunctions $G_n: X \to \mathscr{Z}$ which satisfy, for each $x \in X$, the properties (a_1) $G_n(x) \supset F(x)$, (a_2) $G_1(x) \supset G_2(x) \supset \cdots$, (a_3) $h(G_n(x), F(x)) \to 0$ as $n \to +\infty$. Some refinements are considered as well. For instance (see Theorem 4.8), the multifunctions $F: X \to \mathscr{U}_k(Z)$ which are completely Hausdorff upper semicontinuous are characterized by property $(*)$ in which each multifunction $G_n: X \to \mathscr{U}_k(Z)$ is completely continuous.

Analogous results for certain classes of Hausdorff lower semicontinuous multifunctions $F: X \to \mathscr{Z}$ are established (Theorems 3.1 and 3.6). In this case, the continuous approximations $G_n: X \to \mathscr{Z}$ under consideration satisfy, for each $x \in X$, the properties (a_1) $G_n(x) \subset F(x)$, (a_2) $G_1(x) \subset G_2(x) \subset \cdots$, (a_3) $h(G_n(x), F(x)) \to 0$ as $n \to +\infty$.

2. Notations and Preliminaries

Throughout the paper X and Z will denote, respectively, a metric space and a real normed space. Let 2^Z be the family of all nonempty subsets of Z. We shall consider the following subfamilies of 2^Z:

$$\mathscr{U}(Z) = \{A \in 2^Z \mid A \text{ is convex and bounded}\}$$

$$\mathscr{U}_c(Z) = \{A \in 2^Z \mid A \text{ is convex closed and bounded}\}$$

$$\mathscr{U}_k(Z) = \{A \in 2^Z \mid A \text{ is convex compact}\}$$

$$\mathscr{U}_0(Z) = \{A \in 2^Z \mid A \text{ is convex closed bounded with nonempty interior}\}$$

$$\mathscr{V}(Z) = \{A \in 2^Z \mid A \text{ is bounded}\}.$$

For any metric space $\mathfrak{X}$, we agree to denote by ρ the distance function.

Let $\mathfrak{X}$ be a metric space. For any set $A \subset \mathfrak{X}$, we denote by $\bar{A}$ the closure of A. If $u \in \mathfrak{X}$ and $A \subset \mathfrak{X}$ is nonempty, we put $d(u, A) = \inf\{\rho(u, a) \mid a \in A\}$. For any $u \in \mathfrak{X}$ and $r > 0$, we set $S(u, r) = \{x \in \mathfrak{X} \mid \rho(x, u) < r\}$. Whenever $\mathfrak{X} = Z$

we write, for notational convenience, S in place of $S(0, 1)$ (0 denotes the zero of Z). For each $A \subset Z$, $\overline{\text{co}}\, A$ stands for the closed convex hull of A. For any A, $B \in \mathscr{P}(Z)$ we put

$$h(A, B) = \inf\{t > 0 \mid A \subset B + tS, B \subset A + tS\}.$$

As is well known, h is a pseudometric in $\mathscr{P}(Z)$ and, in particular, in $\mathscr{U}(Z)$. When restricted to $\mathscr{U}_k(Z)$, $\mathscr{U}_c(Z)$ or $\mathscr{U}_0(Z)$, h is the usual Hausdorff distance.

$\mathbb{N}$ and $\mathbb{R}$ stand, respectively, for the positive integers and the real numbers.

Let A, $B \in 2^Z$ and $r \in \mathbb{R}$. As usual, we define $A + B = \{a + b \mid a \in A, b \in B\}$ and $rA = \{ra \mid a \in A\}$. The properties reviewed in the following propositions are known.

PROPOSITION 2.1. *Let A, B, $C \in 2^Z$ and r, $s \in \mathbb{R}$. Then,*

$$A + 0 = A, \qquad 1A = A$$

$$A + B = B + A, \qquad r(sA) = (rs)A$$

$$A + (B + C) = (A + B) + C, \qquad r(A + B) = rA + rB.$$

Furthermore, we have: (a_1) $(r + s)A \subset rA + sA$; (a_2) *if A is convex and r, s are both nonnegative (or nonpositive), then* $(r + s)A = rA + sA$; (a_3) *for any $r \neq 0$, $rA \subset rB$ if and only if $A \subset B$.*

PROPOSITION 2.2 (RÅDSTRÖM [10 p. 167]). *Let A, B, $C \in 2^Z$. Suppose B convex closed, C bounded and that $A + C \subset B + C$. Then $A \subset B$.*

PROPOSITION 2.3. *Let A, B, C, $D \in \mathscr{P}(Z)$ and $r \in \mathbb{R}$. Then,*

(a_1) $h(A + B, C + D) \leqslant h(A, C) + h(B, D)$,

(a_2) $h(rA, rB) = |r|\, h(A, B)$.

Furthermore, if A, $B \in \mathscr{U}(Z)$ and r, $s \in \mathbb{R}$, we have

(a_3) $h(rA, sA) \leqslant |r - s|\, h(A, 0)$,

(a_4) $h(rA, sB) \leqslant |r|\, h(A, B) + |r - s|\, h(B, 0)$.

Proof. (a_1) follows easily from the definition of Hausdorff pseudo-metric.

(a_2) Let $r \neq 0$ (otherwise there is nothing to prove) and let $t > h(rA, rB)$. Then, $rA \subset rB + tS$, $rB \subset rA + tS$ and so, $A \subset B + (t/r)S$, $B \subset A + (t/r)S$, from which $A \subset B + (t/|r|)S$, $B \subset A + (t/|r|)S$ follow, being $(t/r)\,S = (t/|r|)\,S$. Hence, $h(A, B) \leqslant t/|r|$, that is, $|r|\,h(A, B) \leqslant t$ from which, letting $t \to h(rA, rB)$, we obtain $|r|\,h(A, B) \leqslant h(rA, rB)$. To prove the reverse inequality take $t > h(A, B)$. Then, $A \subset B + tS$, $B \subset A + tS$ thus, $rA \subset rB + |r|\,tS$, $rB \subset rA + |r|\,tS$, from which $h(rA, rB) \leqslant |r|\,t$ follows. Letting $t \to h(A, B)$ we obtain $h(rA, rB) \leqslant |r|\,h(A, B)$ and (a_2) is true.

(a_3) Suppose $r \geqslant s \geqslant 0$. Since A is convex (and bounded) we have $h(rA, sA) = h([(r-s)+s]\,A, sA) = h((r-s)\,A + sA,\ 0 + sA) \leqslant h((r-s)\,A, 0) = |r-s|\,h(A, 0)$. Whenever $s \leqslant r \leqslant 0$ the proof is similar. Lastly, if $s < 0 < r$, we have $h(rA, sA) \leqslant h(rA, 0) + h(0, sA) = rh(A, 0) + |s|\,h(A, 0) = |r-s|\,h(A, 0)$. Hence (a_3) is true.

(a_4) From $h(rA, sB) \leqslant h(rA, rB) + h(rB, sB)$, (a_4) follows at once, by virtue of (a_2) and (a_3). The proof is complete.

Proposition 2.4. *Let A and A_n $(n = 1, 2, \ldots)$ be nonempty closed bounded subsets of Z satisfying $A_1 \supset A_2 \supset \cdots$ and $h(A_n, A) \to 0$ as $n \to +\infty$. Then $\bigcap_{n=1}^{\infty} A_n = A$.*

Proof. It is easy to see that for every $n \in \mathbb{N}$ we have $A \subset A_n$, thus $A \subset B = \bigcap_{n=1}^{\infty} A_n$. Suppose that $A \neq B$ and let $b \in B$ be such that $b \notin A$, that is, $d(b, A) > 0$. From $0 < d(b, A) \leqslant d(b, A_n) + h(A_n, A)$ $(n = 1, 2, \ldots)$, observing that $b \in A_n$ and $h(A_n, A) \to 0$ as $n \to +\infty$, a contradiction follows. Thus $A = B$. This completes the proof.

Remark 2.5. If the sets A_n are nonempty compact and satisfy $A_1 \supset A_2 \supset \cdots$ then $h(A_n, A) \to 0$ as $n \to +\infty$ if and only if $\bigcap_{n=1}^{\infty} A_n = A$.

Let Y be a nonempty set contained in X.

A multifunction $F: Y \to \mathscr{C}(Z)$ is said to be *Hausdorff lower semicontinuous*, for short H–l.s.c. (resp. *Hausdorff upper semicontinuous*, for short H–u.s.c.) if for each $u \in Y$ and each $\varepsilon > 0$ there is $\delta = \delta(u, \varepsilon) > 0$ such that $F(u) \subset F(x) + \varepsilon S$ (resp. $F(x) \subset F(u) + \varepsilon S$) whenever $x \in Y \cap S(u, \delta)$. A multifunction which is both H–l.s.c. and H–u.s.c. is called *continuous*.

A multifunction $F: Y \to \mathscr{K}_k(Z)$ which is H–l.s.c. (resp. H–u.s.c., continuous) and satisfies $\bigcup \{F(x) | x \in Y\} \subset K$, where K is a convex compact subset of Z, is called *H–l.s.c.* (resp. *H–u.s.c.*, *continuous*) and *compact*.

A multifunction $F: Y \to \mathscr{K}_k(Z)$ is called *completely H–l.s.c.* (resp. *H–u.s.c.*, *continuous*) if F is H–l.s.c. (resp. H–u.s.c., continuous) and, furthermore, has the property: for each bounded set $B \subset Y$ there is a convex compact set $K \subset Z$ (depending on B) satisfying $\bigcup \{F(x) | x \in B\} \subset K$.

A multifunction $F: Y \to \mathscr{C}(Z)$ is called *bounded* if there is a constant $M \geqslant 0$ such that $h(F(x), 0) \leqslant M$ for each $x \in Y$.

A multifunction $F: Y \to \mathscr{C}(Z)$ is said to be *locally Lipschitzean* if for each $u \in Y$ there are $\delta(u) > 0$ and $L(u) \geqslant 0$ such that for every $x_1, x_2 \in Y \cap S(u, \delta(u))$ we have $h(F(x_1), F(x_2)) \leqslant L(u)\,\rho(x_1, x_2)$.

Let $\mathfrak{X}$ be a metric space. For any function $q: \mathfrak{X} \to \mathbb{R}$, the *support* of q, which we denote by supp q, is the closed set $\overline{\{x \in \mathfrak{X} \mid q(x) \neq 0\}}$. A family $\{q_i\}_{i \in I}$ of continuous functions $q_i: \mathfrak{X} \to [0, 1]$ is called a *partition of unity* on $\mathfrak{X}$ if:

(a_1) the family $\{\operatorname{supp} q_i\}_{i \in I}$ is a neighborhood finite closed covering of $\mathfrak{X}$.

(a_2) $\sum_{i \in I} q_i(x) = 1$ for each $x \in \mathfrak{X}$.

If $\mathscr{V} = \{V_j\}_{j \in J}$ is a given open covering of $\mathfrak{X}$ we say that a partition $\{q_j\}_{j \in J}$ of unity is subordinated to $\mathscr{V}$ if the support of each q_j lies in the corresponding V_j.

As it is well known (see Dugundji [2, p. 170]), for each open covering $\mathscr{V} = \{V_j\}_{j \in J}$ of the metric space $\mathfrak{X}$ there is a partition of unity subordinated to $\mathscr{V}$. Furthermore, the partition of unity can be chosen to consist of locally Lipschitzean functions.

PROPOSITION 2.6. *Let Y be a metric space and let $\mathscr{X}$ denote any of the spaces $\mathscr{U}_k(Z)$, $\mathscr{U}_c(Z)$, $\mathscr{U}_0(Z)$. Let $\mathscr{V} = \{V_j\}_{j \in J}$ be a covering of Y by nonempty open sets $V_j \subset Y$ and suppose that with each $j \in J$ there corresponds a continuous and bounded multifunction $G_j: W_j \to \mathscr{X}$, where $V_j \subset W_j \subset Y$. Let $\mathscr{Q} = \{q_j\}_{j \in J}$ be a partition of unity subordinated to $\mathscr{V}$. Then, if we set*

$$G(x) = \overline{G^*(x)} \qquad \text{where} \qquad G^*(x) = \sum_{j \in J} q_j(x)\, G_j(x), \qquad x \in Y \quad (2.1)$$

we have $G(x) \in \mathscr{X}$ and the multifunction $G: Y \to \mathscr{X}$ defined by (2.1) is continuous. Moreover, if all q_j and G_j $(j \in J)$ are locally Lipschitzean G is also locally Lipschitzean.

Proof. Let $x_0 \in Y$. Since $\{\operatorname{supp} q_j\}_{j \in J}$ is a neighborhood finite closed covering of Y there is $\delta_0 > 0$ such that the set of all $q_j \in \mathscr{Q}$ the supports of which meet $S(x_0, \delta_0)$ is nonempty and finite. Denote the set of these functions by $\{q_i\}_{i \in I}$ where $I = \{1, 2, \ldots, s\}$. Evidently we have

$$G^*(x) = \sum_{i \in I} q_i(x)\, G_i(x) \qquad \text{for each } x \in S(x_0, \delta_0),$$

thus $G^*(x)$ is convex and bounded since so is each $G_i(x)$ and, clearly, $G(x) \in \mathscr{X}$.

Now we shall prove that G is locally Lipschitzean, whenever all q_j and G_j are so. To this end, let $I_0 = \{i \in I \mid x_0 \in \operatorname{supp} q_i\}$. Then, for each $i \in I \setminus I_0$ we have $d(x_0, \operatorname{supp} q_i) > 0$ and so there is $0 < \delta_1 < \delta_0$ such that

$$S(x_0, \delta_1) \cap \operatorname{supp} q_i = \varnothing \qquad \text{for each } i \in I \setminus I_0.$$

Consequently,

$$G^*(x) = \sum_{i\in I_0}' q_i(x)\, G_i(x) \qquad \text{for each} \quad x \in S(x_0, \delta_1). \tag{2.2}$$

For each $i \in I_0$ we have $x_0 \in \operatorname{supp} q_i \subset V_i$; furthermore, G_i is locally Lipschitzean and bounded on the open set $V_i \subset W_i$ and q_i is locally Lipschitzean on Y. Thus, there is $0 < \delta_2 < \delta_1$ such that we have

$$S(x_0, \delta_2) \subset V_i \qquad \text{for each } i \in I_0$$

and, for $i \in I_0$,

$$h(G_i(x_1), G_i(x_2)) \leqslant L\rho(x_1, x_2) \qquad \text{for each } x_1, x_2 \in S(x_0, \delta_2) \tag{2.3}$$

$$|q_i(x_1) - q_i(x_2)| \leqslant M\rho(x_1, x_2) \qquad \text{for each } x_1, x_2 \in S(x_0, \delta_2) \tag{2.4}$$

$$h(G_i(x), 0) \leqslant N \qquad \text{for each } x \in S(x_0, \delta_2), \tag{2.5}$$

where L, M, N are constants independent of $i \in I_0$. Let $x_1, x_2 \in S(x_0, \delta_2)$. Then from (2.2), by Proposition 2.3(a_1), (a_4), we have

$$\begin{aligned} &h(G^*(x_1), G^*(x_2)) \\ &\quad = h\left(\sum_{i\in I_0}' q_i(x_1)\, G_i(x_1), \sum_{i\in I_0}' q_i(x_2)\, G_i(x_2)\right) \\ &\quad \leqslant \sum_{i\in I_0}' h(q_i(x_1)\, G_i(x_1), q_i(x_2)\, G_i(x_2)) \\ &\quad \leqslant \sum_{i\in I_0}' |q_i(x_1)\, h(G_i(x_1), G_i(x_2)) \\ &\qquad + |q_i(x_1) - q_i(x_2)|\, h(G_i(x_2), 0)|. \end{aligned}$$

Taking into account (2.3), (2.4) and (2.5) we obtain

$$\begin{aligned} h(G^*(x_1), G^*(x_2)) &\leqslant \sum_{i\in I_0}' |L\rho(x_1, x_2) + MN\rho(x_1, x_2)| \\ &\leqslant s(L + MN)\, \rho(x_1, x_2), \end{aligned}$$

and G^* is locally Lipschitzean. Clearly, the multifunction G is also locally Lipschitzean. A similar argument can be used to show that G is continuous whenever all G_j are continuous. This completes the proof.

3. Continuous Approximations of Lower Semicontinuous Multifunctions

Theorem 3.1. *Let a multifunction $F: X \to \mathscr{U}_k(Z)$ be given, where Z is a separable real Banach space. Then the following two statements are equivalent:*

(a) *F is H–l.s.c.*

(b) *There exists a sequence $\{G_n\}$ of continuous multifunctions $G_n: X \to \mathscr{U}_k(Z)$ satisfying, for each $x \in X$, the properties:* (b_1) $G_n(x) \subset F(x)$ $(n \in \mathbb{N})$, (b_2) $G_1(x) \subset G_2(x) \subset \cdots$, ($b_3$) $h(G_n(x), F(x)) \to 0$ *as* $n \to +\infty$.

Proof. (b) $\Rightarrow$ (a) Let $x_0 \in X$. Let $\varepsilon > 0$. By (b_3) there is $n_0 \in \mathbb{N}$ such that $F(x_0) \subset G_{n_0}(x_0) + \varepsilon S$. Since G_{n_0} is continuous, there is $\delta > 0$ such that $G_{n_0}(x_0) \subset G_{n_0}(x) + \varepsilon S$ for each $x \in S(x_0, \delta)$. Hence,

$$F(x_0) \subset G_{n_0}(x) + 2\varepsilon S \qquad \text{for each } x \in S(x_0, \delta)$$

and so, because of (b_1), we have

$$F(x_0) \subset F(x) + 2\varepsilon S \qquad \text{for each } x \in S(x_0, \delta),$$

from which it follows that F is H–l.s.c.

(a) $\Rightarrow$ (b) Since any H–l.s.c. multifunction is certainly lower semicontinuous (according to the definition used by Michael [9, p. 362]), by a result of Michael [9, Lemma 5.2] there is an (infinite) sequence of continuous selections $f_n: X \to Z$ of F satisfying, for each $x \in X$,

$$F(x) = \overline{\bigcup_{n \in \mathbb{N}} \{f_n(x)\}}.$$

For each $n \in \mathbb{N}$, let $G_n: X \to \mathscr{U}_k(Z)$ be the multifunction defined by

$$G_n(x) = \overline{\mathrm{co}}\,\{f_1(x), f_2(x), \ldots, f_n(x)\}, \qquad x \in X.$$

Clearly, G_n is continuous and satisfies properties (b_1) and (b_2). For a contradiction, suppose that $\{G_n\}$ does not satisfy (b_3). There are then $x_0 \in X$ and $\varepsilon > 0$ such that, for each $n \in \mathbb{N}$, $h(G_n(x_0), F(x_0)) > \varepsilon$ and so $F(x_0) \not\subset G_n(x_0) + \varepsilon S$. Consequently, there is a sequence $\{a_n\} \subset F(x_0)$ such that

$$a_n \notin G_n(x_0) + \varepsilon S \qquad \text{for each } n \in \mathbb{N}.$$

Since $F(x_0)$ is compact, a subsequence, say $\{a_n\}$, will converge to a point $a \in F(x_0)$ thus, taking n large enough ($n \geqslant n_0$), we have $|a_n - a| < \varepsilon/2$. Then we have

$$d(a, G_n(x_0)) \geqslant d(a_n, G_n(x_0)) - |a_n - a| > \varepsilon - \varepsilon/2 = \varepsilon/2 \qquad \text{for each } n \geqslant n_0. \tag{3.1}$$

On the other hand, $\{f_n(x_0)\}_{n \geqslant n_0}$ is also dense in $F(x_0)$ (for $F(x_0)$ is convex) and so there is $n_1 \geqslant n_0$ such that $|f_{n_1}(x_0) - a| < \varepsilon/2$. This implies that $d(a, G_{n_1}(x_0)) < \varepsilon/2$, which contradicts (3.1). Thus (b_3) is true. This completes the proof.

Remark 3.2. More generally, the implication (b) $\Rightarrow$ (a) is true if, in the statement of Theorem 3.1, Z is a real normed space and the multifunctions G_n ($n \in \mathbb{N}$) are supposed to be H-l.s.c. instead of continuous (see [6]). Moreover, under these assumptions, (b) $\Rightarrow$ (a) is true even with $\mathscr{U}_c(Z)$ or $\mathscr{U}_0(Z)$ in the place of $\mathscr{U}_k(Z)$.

LEMMA 3.3. *Let $A \in \mathscr{U}_0(Z)$ and let $v \in A$ be such that $v + \theta S \subset A$ for some $\theta > 0$. For each $0 < r < 1$ put $A_r = rv + (1-r)A$. Then A_r is in $\mathscr{U}_0(Z)$ and has the properties*: (a_1) $A_r + r\theta S \subset A$ *and* (a_2) $h(A_r, A) \to 0$ *as* $r \to 0$.

Proof. Since $A_r \supset rv + (1-r)(v + \theta S) = v + (1-r)\theta S$ and, clearly, A_r is convex closed and bounded, it follows that $A_r \in \mathscr{U}_0(Z)$ $(0 < r < 1)$. Moreover, A is convex and contains $v + \theta S$ thus $A \supset r(v + \theta S) + (1-r)A = A_r + r\theta S$ and (a_1) is true. From

$$\begin{aligned} h(A_r, A) &= h(rv + (1-r)A, rA + (1-r)A) \\ &\leqslant h(rv, rA) = rh(v, A), \end{aligned}$$

letting $r \to 0$, (a_2) follows. This completes the proof.

PROPOSITION 3.4. *Let $F: X \to \mathscr{U}_0(Z)$ be an H-l.s.c multifunction. Then there exists a sequence $\{H_n\}$ of continuous multifunctions $H_n: X \to \mathscr{U}_0(Z)$ satisfying, for each $x \in X$, the properties*: (a_1) $H_n(x) \subset F(x)$ ($n \in \mathbb{N}$) *and* (a_2) $h(H_n(x), F(x)) \to 0$ *as* $n \to +\infty$.

Proof. Set $\varepsilon_n = 1/2^n$, $n \in \mathbb{N}$. Let $u \in X$. $F(u) \in \mathscr{U}_0(Z)$ and so, by Lemma 3.3, there exist sequences $\{Q_u^n\}$, $Q_u^n \in \mathscr{U}_0(Z)$, and $\{r_u^n\}$, $r_u^n > 0$, such that $Q_u^n + r_u^n S \subset F(u)$ and

$$h(Q_u^n, F(u)) < \varepsilon_n \qquad \text{for each } n \in \mathbb{N}. \tag{3.2}$$

Let $n \in \mathbb{N}$ be any. Since F is H–l.s.c., there is $0 < \delta_u^n < 1/n$ such that $F(u) \subset F(x) + r_u^n S$ whenever $x \in S(u, \delta_u^n)$. Therefore, $Q_u^n + r_u^n S \subset F(u) \subset F(x) + r_u^n S$ from which, by Proposition 2.2., we have

$$Q_u^n \subset F(x) \qquad \text{for each } x \in S(u, \delta_u^n). \tag{3.3}$$

For every (fixed) $n \in \mathbb{N}$ the family, $\mathscr{S}_n = \{S_u^n\}_{u \in X}$, where $S_u^n = S(u, \delta_u^n)$, is an open covering of X and hence there is a partition $\mathscr{Q}_n = \{q_u^n\}_{u \in X}$ of unity subordinated to $\mathscr{S}_n$. Now, set

$$H_n(x) = \overline{H_n^*(x)} \qquad \text{where} \qquad H_n^*(x) = \sum_{u \in X} q_u^n(x)\, Q_u^n, \qquad x \in X. \tag{3.4}$$

By Proposition 2.6, $H_n(x) \in \mathscr{U}_0(Z)$ and the multifunction $H_n\colon X \to \mathscr{U}_0(Z)$, defined by (3.4), is continuous.

Let $x_0 \in X$. Since $\{\operatorname{supp} q_u^n\}_{u \in X}$ (n fixed) is a neighborhood finite closed covering of X, the set of all $q_u^n \in \mathscr{Q}_n$ the supports of which contain x_0 is nonempty and finite. Denote this set by $\{q_{u_i^n}^n\}_{i=1}^{s_n}$ ($u_i^n \in X$). Evidently,

$$H_n^*(x_0) = \sum_{i=1}^{s_n} q_{u_i^n}^n(x_0)\, Q_{u_i^n}^n. \tag{3.5}$$

For each $1 \leqslant i \leqslant s_n$ we have $x_0 \in \operatorname{supp} q_{u_i^n}^n \subset S(u_i^n, \delta_{u_i^n}^n)$ thus, by (3.3), $Q_{u_i^n}^n \subset F(x_0)$ and hence

$$H_n^*(x_0) \subset \sum_{i=1}^{s_n} q_{u_i^n}^n(x_0)\, F(x_0) = F(x_0).$$

Therefore, $H_n(x_0) \subset F(x_0)$. Since $x_0 \in X$ and $n \in \mathbb{N}$ are arbitrary, (a_1) is satisfied.

Next, let us show that $h(H_n^*(x_0), F(x_0)) \to 0$ as $n \to +\infty$. Supposing the contrary, there is $\varepsilon > 0$ and a subsequence, say, $\{h(H_n^*(x_0), F(x_0))\}$, such that $h(H_n^*(x_0), F(x_0)) > \varepsilon$ for each $n \in \mathbb{N}$. This implies that, for each $n \in \mathbb{N}$, $F(x_0) \not\subset H_n^*(x_0) + \varepsilon S$ and so, in view of (3.5),

$$F(x_0) \not\subset \sum_{i=1}^{s_n} q_{u_i^n}^n(x_0)\, Q_{u_i^n}^n + \varepsilon S = \sum_{i=1}^{s_n} q_{u_i^n}^n(x_0) [Q_{u_i^n}^n + \varepsilon S]. \tag{3.6}$$

From (3.6) it follows that, for every $n \in \mathbb{N}$, there is at least one index, say, i_n ($1 \leqslant i_n \leqslant s_n$), such that

$$F(x_0) \not\subset Q_{u_{i_n}^n}^n + \varepsilon S, \qquad n = 1, 2, \ldots. \tag{3.7}$$

Observe that $u_{i_n}^n \to x_0$ as $n \to +\infty$ because $x_0 \in S(u_{i_n}^n, \delta_{u_{i_n}^n}^n) \subset S(u_{i_n}^n, 1/n)$, $n \in \mathbb{N}$. Since $u_{i_n}^n \to x_0$ as $n \to +\infty$ and F is H–l.s.c, there is $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have $F(x_0) \subset F(u^n_{i_n}) + (\varepsilon/2)\,S$ and so, by virtue of (3.2), $F(x_0) \subset Q^n_{u^n_{i_n}} + (\varepsilon_n + \varepsilon/2)\,S$. Thus, for $n \geqslant n_0$ large enough, we have $F(x_0) \subset Q^n_{u^n_{i_n}} + \varepsilon S$, a contradiction to (3.7). Therefore $h(H^*_n(x_0), F(x_0)) \to 0$ as $n \to +\infty$, which implies that $h(H_n(x_0), F(x_0)) \to 0$ as $n \to +\infty$. Since $x_0 \in X$ is arbitrary, (a_2) is true. This completes the proof.

Remark 3.5. In the statement of Proposition 3.4 one can obtain something more, namely, that the sequence $\{H_n\}$ consists of locally Lipschitzean multifunctions. To this end take, in the above proof, for each $n \in \mathbb{N}$, a partition $\mathscr{Q}_n$ of unity (subordinated to $\mathscr{S}_n$) consisting of locally Lipschitzean functions. Then, Proposition 2.6 implies that each H_n is locally Lipschitzean.

THEOREM 3.6. *Let a multifunction $F: X \to \mathscr{U}_0(Z)$ be given. Then the following two statements are equivalent*:

(a) *F is H–l.s.c.*

(b) *There exists a sequence $\{G_n\}$ of continuous multifunctions $G_n: X \to \mathscr{U}_0(Z)$ satisfying, for each $x \in X$, the properties*: (b_1) $G_n(x) \subset F(x)$ $(n \in \mathbb{N})$, (b_2) $G_1(x) \subset G_2(x) \subset \cdots$, *and* (b_3) $h(G_n(x), F(x)) \to 0$ *as* $n \to +\infty$.

Proof. (b) $\Rightarrow$ (a) This is a consequence of Remark 3.2.

(a) $\Rightarrow$ (b) By virtue of Proposition 3.4, there is a sequence $\{H_n\}$ of multifunctions $H_n: X \to \mathscr{U}_0(Z)$ which are continuous and satisfy properties (a_1) and (a_2). For each $n = 1, 2, \dots$, set

$$G_n(x) = \overline{\mathrm{co}}\, [H_1(x) \cup H_2(x) \cup \cdots \cup H_n(x)], \qquad x \in X. \tag{3.8}$$

Clearly, (3.8) defines a continuous multifunction $G_n: X \to \mathscr{U}_0(Z)$ and $\{G_n\}$ satisfies (b_1) and (b_2). Furthermore, for each $n \in \mathbb{N}$ and $x \in X$, we have $H_n(x) \subset G_n(x) \subset F(x)$ thus $h(G_n(x), F(x)) \leqslant h(H_n(x), F(x))$ from which, letting $n \to +\infty$, (b_3) follows. This completes the proof.

Remark 3.7. If each H_n $(n \in \mathbb{N})$ is chosen to be locally Lipschitzean (which is certainly possible in view of Remark 3.5) then, in the statement of Theorem 3.6, each G_n $(n \in \mathbb{N})$ is also locally Lipschitzean.

4. Continuous Approximations of Upper Semicontinuous Multifunctions

To prove the next proposition we use some ideas introduced by Hukuhara in [6] and technical details due to Lasry and Robert [8] and Haddad [4].

PROPOSITION 4.1. *Let Y be a metric space and let $\mathscr{Z}$ denote any of the spaces $\mathscr{U}_c(Z)$, $\mathscr{U}_0(Z)$. Let $F\colon Y \to \mathscr{Z}$ be H–u.s.c. and bounded, that is, $h(F(x), 0) \leqslant M$ $(M \geqslant 0)$ for each $x \in Y$. Then there is a sequence $\{G_n\}$ of continuous multifunctions $G_n\colon Y \to \mathscr{Z}$ satisfying, for each $x \in Y$, the properties:* (a$_1$) $G_n(x) \subset \Omega = \overline{\mathrm{co}} \bigcup \{F(v) | v \in Y\}$ $(n \in \mathbb{N})$, (a$_2$) $G_n(x) \supset F(x)$ $(n \in \mathbb{N})$, (a$_3$) $G_1(x) \supset G_2(x) \supset \cdots$, *and* (a$_4$) $h(G_n(x), F(x)) \to 0$ *as* $n \to +\infty$.

Proof. For each $u \in Y$ consider the sequence $\{G_u^n\}$, where

$$G_u^n = \overline{\mathrm{co}} \bigcup \{F(x) | x \in S(u, 2/3^n)\}, \qquad n \in \mathbb{N}. \tag{4.1}$$

G_u^n is bounded since F is so and, clearly, $G_u^n \in \mathscr{Z}$. Set $S_u^n = S(u, 1/3^n)$, $u \in Y$, $n \in \mathbb{N}$. For each (fixed) $n \in \mathbb{N}$, $\mathscr{S}_n = \{S_u^n\}_{u \in Y}$ is an open covering of Y and hence there is a partition $\mathscr{Q}_n = \{q_u^n\}_{u \in Y}$ of unity subordinated to $\mathscr{S}_n$. Let us put

$$G_n(x) = \overline{G_n^*(x)} \qquad \text{where} \qquad G_n^*(x) = \sum_{u \in Y} q_u^n(x)\, G_u^n, \qquad x \in Y. \tag{4.2}$$

By Proposition 2.6, $G_n(x) \in \mathscr{Z}$ and the multifunction $G_n\colon Y \to \mathscr{Z}$, defined by (4.2), is continuous.

Let $x_0 \in Y$. Since, for fixed $n \in \mathbb{N}$, $\{\mathrm{supp}\, q_u^n\}_{u \in Y}$ is a neighborhood finite closed covering of Y, the set of all $q_u^n \in \mathscr{Q}_n$ the supports of which contain x_0 is nonempty and finite. Denote this set by $\{q_{u_i^n}^n\}_{i=1}^{s_n}$ $(u_i^n \in Y)$. Evidently,

$$G_n^*(x_0) = \sum_{i=1}^{s_n} q_{u_i^n}^n(x_0)\, G_{u_i^n}^n. \tag{4.3}$$

We are going to prove that $\{G_n\}$ satisfies properties (a$_1$)–(a$_4$).

(a$_1$) Since $G_{u_i^n}^n \subset \Omega$, from (4.3) we obtain $G_n^*(x_0) \subset \Omega$ and so $G_n(x_0) \subset \Omega$. Since $x_0 \in Y$ and $n \in \mathbb{N}$ are arbitrary, (a$_1$) is true.

(a$_2$) For each $1 \leqslant i \leqslant s_n$ we have $x_0 \in \mathrm{supp}\, q_{u_i^n}^n \subset S(u_i^n, 1/3^n)$ thus by (4.1), $G_{u_i^n}^n \supset F(x_0)$. Therefore, from (4.3), $G_n^*(x_0) \supset F(x_0)$ and, a fortiori, $G_n(x_0) \supset F(x_0)$. Since $x_0 \in Y$ and $n \in \mathbb{N}$ are arbitrary, (a$_2$) is satisfied.

(a$_3$) Denote by $\{q_{v_j^{n+1}}^{n+1}\}_{j=1}^{s_{n+1}}$ the (nonempty) finite set of all functions $q_v^{n+1} \in \mathscr{Q}_{n+1}$ the supports of which contain x_0. Then

$$G_{n+1}^*(x_0) = \sum_{j=1}^{s_{n+1}} q_{v_j^{n+1}}^{n+1}(x_0)\, G_{v_j^{n+1}}^{n+1}. \tag{4.4}$$

Let $1 \leqslant j \leqslant s_{n+1}$ and $1 \leqslant i \leqslant s_n$ by any. We shall see that $G_{v_j^{n+1}}^{n+1} \subset G_{u_i^n}^n$. First, observe that $x_0 \in \mathrm{supp}\, q_{v_j^{n+1}}^{n+1}$, $x_0 \in \mathrm{supp}\, q_{u_i^n}^n$ thus

$$x_0 \in S\left(v_j^{n+1}, \frac{1}{3^{n+1}}\right), \qquad x_0 \in S\left(u_i^n, \frac{1}{3^n}\right).$$

Then, any $x \in S(v_j^{n+1}, 2/3^{n+1})$ satisfies $\rho(x, u_i^n) \leqslant \rho(x, v_j^{n+1}) + \rho(v_j^{n+1}, x_0) + \rho(x_0, u_i^n) < 2/3^{n+1} + 1/3^{n+1} + 1/3^n = 2/3^n$ and hence $S(v_j^{n+1}, 2/3^{n+1}) \subset S(u_i^n, 2/3^n)$. This implies that $G_{v_j^{n+1}}^{n+1} \subset G_{u_i^n}^n$ and so

$$G_{v_j^{n+1}}^{n+1} = \sum_{i=1}^{s_n} q_{u_i^n}^n(x_0)\, G_{v_j^{n+1}}^{n+1} \subset \sum_{i=1}^{s_n} q_{u_i^n}^n(x_0)\, G_{u_i^n}^n = G_n^*(x_0).$$

Thus, from (4.4),

$$G_{n+1}^*(x_0) = \sum_{j=1}^{s_{n+1}} q_{v_j^{n+1}}^{n+1}(x_0)\, G_{v_j^{n+1}}^{n+1} \subset \sum_{j=1}^{s_{n+1}} q_{v_j^{n+1}}^{n+1}(x_0)\, G_n^*(x_0) = G_n^*(x_0)$$

from which $G_{n+1}(x_0) \subset G_n(x_0)$ follows. Since $x_0 \in Y$ and $n \in \mathbb{N}$ are arbitrary, (a_3) is proved.

(a_4) Suppose that (a_4) is false (for $x = x_0$). This implies that, for some $\varepsilon > 0$, we have $h(G_n^*(x_0), F(x_0)) > \varepsilon$ $(n \in \mathbb{N})$, and so $G_n^*(x_0) \not\subset F(x_0) + \varepsilon S$ for each $n \in \mathbb{N}$. We retain the notation introduced in the first part of the proof, thus for each $n \in \mathbb{N}$, $G_n^*(x_0)$ is expressed by (4.3). If for any $n \in \mathbb{N}$ and every $1 \leqslant i \leqslant s_n$ the inclusion $G_{u_i^n}^n \subset F(x_0) + \varepsilon S$ were satisfied we would obtain from (4.3) $G_n^*(x_0) \subset F(x_0) + \varepsilon S$, a contradiction. Hence, for every $n \in \mathbb{N}$, there is some $u_{i_n}^n$, $1 \leqslant i_n \leqslant s_n$, for which $G_{u_{i_n}^n}^n \not\subset F(x_0) + \varepsilon S$ and, in view of (4.1), there is also $x_n \in S(u_{i_n}^n, 2/3^n)$ such that

$$F(x_n) \not\subset F(x_0) + \varepsilon S \qquad \text{for each } n \in \mathbb{N}. \tag{4.5}$$

Since $x_0 \in \operatorname{supp} q_{u_{i_n}^n}^n \subset S(u_{i_n}^n, 1/3^n)$, we have $\rho(x_n, x_0) \leqslant \rho(x_n, u_{i_n}^n) + \rho(u_{i_n}^n, x_0)$ and so $x_n \to x_0$ as $n \to +\infty$. From this and (4.5) it follows that F is not H–u.s.c., a contradiction. Hence (a_4) is true. This completes the proof.

Remark 4.2. In the statement of Proposition 4.1 the sequence $\{G_n\}$ can be chosen to consist of locally Lipschitzean multifunctions. To this end, for each $n \in \mathbb{N}$, take a partition $\mathscr{Q}_n$ of unity (subordinated to $\mathscr{S}_n$) consisting of locally Lipschitzean functions. Then, by virtue of Proposition 2.6, it follows that each G_n $(n \in \mathbb{N})$ is locally Lipschitzean.

Remark 4.3. In Proposition 4.1 let $\mathscr{X} = \mathscr{U}_k(Z)$ and, in addition to all other hypotheses, suppose that $F\colon Y \to \mathscr{U}_k(Z)$ is compact. Then, by the same argument of Proposition 4.1, one has that there is a sequence $\{G_n\}$ of continuous compact multifunctions $G_n\colon Y \to \mathscr{U}_k(Z)$ satisfying properties (a_1)–(a_4). Moreover, as explained in Remark 4.2, we can have that each G_n also be locally Lipschitzean.

PROPOSITION 4.4. *Let $F\colon X \to \mathscr{X}$ be an H–u.s.c. multifunction, where $\mathscr{X}$ denotes any of the spaces $\mathscr{U}_k(Z)$, $\mathscr{U}_c(Z)$, $\mathscr{U}_0(Z)$. Let $\mathscr{V} = \{V_j\}_{j \in J}$ be a* *covering of X by nonempty open sets $V_j \subset X$ and suppose that with each $j \in J$ there corresponds a sequence $\{G_n^j\}$ of continuous and bounded multifunctions $G_n^j\colon W_j \to \mathcal{X}$, where $V_j \subset W_j \subset X$, satisfying, for each $x \in W_j$, the properties: (a_1) $G_n^j(x) \subset \Omega_j = \overline{\mathrm{co}} \bigcup\{F(v) \| v \in W_j\}$ $(n \in \mathbb{N})$, (a_2) $G_n^j(x) \supset F(x)$ $(n \in \mathbb{N})$, (a_3) $G_1^j(x) \supset G_2^j(x) \supset \cdots$, and (a_4) $h(G_n^j(x), F(x)) \to 0$ as $n \to +\infty$. Let $\mathcal{Q} = \{q_j\}_{j \in J}$ be a partition of unity subordinated to $\mathcal{F}$. For any fixed $n \in \mathbb{N}$, set*

$$G_n(x) = \overline{G_n^*(x)} \qquad \text{where} \qquad G_n^*(x) = \sum_{j \in J} q_j(x)\, G_n^j(x), \qquad x \in X. \tag{4.6}$$

Then $G_n(x) \in \mathcal{X}$ and, for each $n \in \mathbb{N}$, the multifunction $G_n\colon X \to \mathcal{X}$ defined by (4.6) is continuous and satisfies, for each $x \in X$, the properties: (b_1) $G_n(x) \subset \Omega = \overline{\mathrm{co}} \bigcup\{F(v) \| v \in X\} (n \in \mathbb{N})$, (b_2) $G_n(x) \supset F(x)$ $(n \in \mathbb{N})$, (b_3) $G_1(x) \supset G_2(x) \supset \cdots$, and (b_4) $h(G_n(x), F(x)) \to 0$ as $n \to +\infty$. Moreover, if all q_j and G_n $(j \in J,\ n \in \mathbb{N})$ are locally Lipschitzean so is each G_n $(n \in \mathbb{N})$.

Proof. By virtue of Proposition 2.6, for each $n \in \mathbb{N}$, G_n is a well-defined continuous multifunction from X to $\mathcal{X}$; furthermore, G_n is locally Lipschitzean if all q_j and G_n^j $(j \in J,\ n \in \mathbb{N})$ are so. Let us prove that $\{G_n\}$ satisfies properties (b_1)–(b_4).

Let $x_0 \in X$ and, as usual, let $\{q_i\}_{i=1}^s$ be the (nonempty) finite set of all functions $q_j \in \mathcal{Q}$ the supports of which contain x_0. Evidently, for each $n \in \mathbb{N}$, we have

$$G_n^*(x_0) = \sum_{i=1}^{s} q_i(x_0)\, G_n^i(x_0). \tag{4.7}$$

From (4.7), by virtue of (a_1), we obtain

$$G_n^*(x_0) \subset \sum_{i=1}^{s} q_i(x_0)\, \Omega_i \subset \sum_{i=1}^{s} q_i(x_0)\, \Omega = \Omega,$$

and so $G_n(x_0) \subset \Omega$. Since $x_0 \in X$ and $n \in \mathbb{N}$ are arbitrary, (b_1) is satisfied. A similar argument shows also that properties (b_2) and (b_3) are fulfilled. As far as (b_4) is concerned, we have

$$\begin{aligned} h(G_n(x_0), F(x_0)) &= h\left(\sum_{i=1}^{s} q_i(x_0)\, G_n^i(x_0), \sum_{i=1}^{s} q_i(x_0)\, F(x_0)\right) \\ &\leqslant \sum_{i=1}^{s} h(q_i(x_0)\, G_n^i(x_0), q_i(x_0)\, F(x_0)) \\ &\leqslant \sum_{i=1}^{s} q_i(x_0)\, h(G_n^i(x_0), F(x_0)). \end{aligned}$$

By (a_4), the last quantity vanishes as $n \to +\infty$ and so, since $x_0 \in X$ is arbitrary, (b_4) is satisfied. This completes the proof.

THEOREM 4.5. *Let a multifunction $F: X \to \mathscr{Z}$ be given, where $\mathscr{Z}$ denotes any of the spaces $\mathscr{U}_c(Z)$, $\mathscr{U}_0(Z)$. Then the following two statements are equivalent:*

(a) *F is H–u.s.c.*

(b) *There exists a sequence $\{G_n\}$ of continuous multifunctions $G_n: X \to \mathscr{Z}$ satisfying, for each $x \in X$, the properties:* (b_1) $G_n(x) \subset \Omega = \overline{\mathrm{co}} \bigcup \{F(v) | v \in X\}$ $(n \in \mathbb{N})$, (b_2) $G_n(x) \supset F(x)$ $(n \in \mathbb{N})$, (b_3) $G_1(x) \supset G_2(x) \supset \cdots$, *and* (b_4) $h(G_n(x), F(x)) \to 0$ *as* $n \to +\infty$.

Proof. (b) $\Rightarrow$ (a) Let $x_0 \in X$ and $\varepsilon > 0$. By (b_4) there is $n_0 \in \mathbb{N}$ such that $G_{n_0}(x_0) \subset F(x_0) + \varepsilon S$. Since G_{n_0} is continuous there is $\delta > 0$ such that $G_{n_0}(x) \subset G_{n_0}(x_0) + \varepsilon S$ for each $x \in S(x_0, \delta)$. Hence

$$G_{n_0}(x) \subset F(x_0) + 2\varepsilon S \qquad \text{for each } x \in S(x_0, \delta)$$

and so, because of (b_2), we have

$$F(x) \subset F(x_0) + 2\varepsilon S \qquad \text{for each } x \in S(x_0, \delta),$$

from which it follows that F is H–u.s.c.

(a) $\Rightarrow$ (b) Since F is H–u.s.c., for each $u \in X$ there is $\delta(u) > 0$ such that $F(x) \subset F(u) + S$ whenever $x \in S(u, \delta(u))$. Set $S_u = S(u, \delta(u))$ and denote by F^u the restriction of F to S_u. For each $u \in X$, $F^u: S_u \to \mathscr{Z}$ is H–u.s.c. and bounded and hence, by Proposition 4.1 (with S_u and F^u in the place of Y and F, respectively), there is a sequence $\{G_n^u\}$ of continuous (bounded) multifunctions $G_n^u: S_u \to \mathscr{Z}$ satisfying properties (a_1)–(a_4) of Proposition 4.1. Let $\mathscr{Q} = \{q_u\}_{u \in X}$ be partition of unity subordinated to the open covering $\mathscr{S} = \{S_u\}_{u \in X}$ of X. For any $n \in \mathbb{N}$, set

$$G_n(x) = \overline{G_n^*(x)} \qquad \text{where} \qquad G_n^*(x) = \sum_{u \in X} q_u(x)\, G_n^u(x), \qquad x \in X. \quad (4.8)$$

From Proposition 4.4 (with $\mathscr{S}$ and G_n^u in the place of $\mathscr{F}$ and G_n^j, respectively), it follows that, for each $n \in \mathbb{N}$, we have $G_n(x) \in \mathscr{Z}$ and the multifunction $G_n: X \to \mathscr{Z}$, defined by (4.8), is continuous; moreover, $\{G_n\}$ satisfies properties (b_1)–(b_4) of the statement. This completes the proof.

Remark 4.6. The implication (b) $\Rightarrow$ (a) remains true if we let $\mathscr{Z} = \mathscr{U}_k(Z)$. Moreover, if $\mathscr{Z}$ stands for any of the spaces $\mathscr{U}_k(Z)$, $\mathscr{U}_c(Z)$, $\mathscr{U}_0(Z)$

the implication (b) $\Rightarrow$ (a) is still valid if the multifunctions G_n ($n \in \mathbb{N}$) are supposed to be H–u.s.c. instead of continuous (see [6]).

Remark 4.7. In the proof of (a) $\Rightarrow$ (b) we can take, in view of Remark 4.2, the sequences $\{G_n^u\}$ $(u \in X)$ consisting of locally Lipschitzean multifunctions. Then, by choosing a partition $\mathscr{Q}$ of unity (subordinated to $\mathscr{S}$) consisting of locally Lipschitzean functions, it follows that each G_n ($n \in \mathbb{N}$) is locally Lipschitzean as well.

THEOREM 4.8. *Let a multifunction $F: X \to \mathscr{U}_k(Z)$ be given. Then the following two statements are equivalent*:

(a) *F is completely H–u.s.c.*

(b) *There exists a sequence $\{G_n\}$ of completely continuous multifunctions $G_n: X \to \mathscr{U}_k(Z)$ satisfying, for each $x \in X$, the properties* (b_1)–(b_4) *of Theorem* 4.5.

Proof. (b) $\Rightarrow$ (a) By Remark 4.6, F is H–u.s.c. Furthermore, for any nonempty bounded set $B \subset X$ we have $\bigcup\{F(x) | x \in B\} \subset \bigcup\{G_1(x) | x \in B\}$ and so F is completely H–u.s.c., for G_1 is completely continuous.

(a) $\Rightarrow$ (b) Fix a point $x_0 \in X$ and consider the sequence $\{C_r\}$ of sets $C_r \subset X$ where $C_1 = S(x_0, 2)$, $C_r = S(x_0, r+1) \backslash \overline{S(x_0, r-1)}$ ($r = 2, 3,\ldots$). By suppressing all sets C_r which are empty, we obtain a new (perhaps finite) sequence $\mathscr{S} = \{C_{r_j}\}$ consisting of nonempty open sets C_{r_j}. Evidently, $\mathscr{S}$ is an open covering of X. Denote by F^j the restriction of F to C_{r_j}. For each j, F^j is H–u.s.c. and compact and so, by Remark 4.3, there is a sequence $\{G_n^j\}$ of continuous compact multifunctions $G_n^j: C_{r_j} \to \mathscr{U}_k(Z)$ satisfying properties (a_1)–(a_4) of Proposition 4.1 (with C_{r_j}, F^j, Ω_j in the place of Y, F, Ω, respectively). Next, let $\mathscr{Q} = \{q_j\}_{j \in J}$ be a partition of unity subordinated to $\mathscr{S}$ and, for each $n \in \mathbb{N}$, set

$$G_n(x) = \sum_{j \in J} q_j(x)\, G_n^j(x), \qquad x \in X. \tag{4.9}$$

Notice that $G_n(x) \in \mathscr{U}_k(Z)$. From Proposition 4.4 (with $\mathscr{Z} = \mathscr{U}_k(Z)$, $V_j = W_j = C_{r_j}$ and $\mathscr{F} = \mathscr{S}$) it follows that, for each $n \in \mathbb{N}$, the multifunction $G_n: X \to \mathscr{U}_k(Z)$, defined by (4.9) is continuous; moreover, the sequence $\{G_n\}$ satisfies, for each $x \in X$, the properties (b_1)–(b_4). To conclude the proof we need to show that each G_n is completely continuous.

In fact let $B \subset X$ be a nonempty bounded set and let $s \in \mathbb{N}$ be such that $B \subset S(x_0, r_s)$. Then $C_{r_j} \cap B = \varnothing$ for every $j \geqslant s+1$ and, since each q_j has support which is contained in C_{r_j}, it follows that $q_j(x) = 0$ for each $x \in B$, whenever $j \geqslant s+1$. Thus, for every $x \in B$,

$$G_n(x) = \sum_{j=1}^{s} q_j(x)\, G_n^j(x) \subset \sum_{j=1}^{s} q_j(x)\, \Omega_j \subset \sum_{j=1}^{s} q_j(x)\, \tilde{\Omega} = \tilde{\Omega},$$

where $\Omega_j = \overline{\mathrm{co}} \bigcup \{F(v) | v \in C_{r_j}\}$ and $\tilde{\Omega} = \overline{\mathrm{co}} \bigcup \{F(v) | v \in S(x_0, r_s + 1)\}$. Then $\bigcup \{G_n(x) | x \in B\} \subset \tilde{\Omega}$ and, since $\tilde{\Omega}$ is convex compact, (a) $\Rightarrow$ (b) is proved. This completes the proof.

Remark 4.9. In the proof of (a) $\Rightarrow$ (b) we can take, by virtue of Remark 4.3, all sequences $\{G_n^j\}$ consisting of multifunctions G_n^j which are continuous compact and locally Lipschitzean. Then, by choosing a partition $\mathscr{S}$ of unity consisting of locally Lipschitzean functions we have, by virtue of Proposition 2.6, that the sequence $\{G_n\}$ of the statement consists of multifunctions which are also locally Lipschitzean.

Remark 4.10. By Proposition 2.4, it follows that the sequence $\{G_n\}$ in the statement of Theorem 4.5 (and Theorem 4.8) satisfies, for each $x \in X$,

$$\bigcap_{n=1}^{\infty} G_n(x) = F(x).$$

5. A Between Theorem for Semicontinuous Multifunctions

The following result is a multivalued version of a theorem proved by Dieudonné [1, Theorem 9] for real-valued functions.

Theorem 5.1. *Let the multifunctions $F: X \to \mathscr{K}_c(Z)$ and $G: X \to \mathscr{K}_0(Z)$ be given. Suppose that F is H–u.s.c., G is H–l.s.c. and that there is a strictly positive function $e: X \to \mathbb{R}$ such that*

$$F(x) + e(x)\, S \subset G(x) \qquad \textit{for each } x \in X.$$

Then there exists a continuous multifunction $H: X \to \mathscr{K}_0(Z)$ and a strictly positive continuous function $r: X \to \mathbb{R}$ satisfying

$$F(x) + r(x)\, S \subset H(x) \qquad \textit{for each } x \in X \tag{5.1}$$

$$H(x) + r(x)\, S \subset G(x) \qquad \textit{for each } x \in X. \tag{5.2}$$

In particular, $F(x) \subset H(x) \subset G(x)$, $x \in X$.

Proof. Let $u \in X$. Since F is H–u.s.c. and G is H–l.s.c. there is $\delta(u) > 0$ such that

$$F(x) \subset F(u) + \frac{e(u)}{4} S, \qquad G(u) \subset G(x) + \frac{e(u)}{4} S \qquad \text{for each } x \in S_u, \quad (5.3)$$

where $S_u = S(u, \delta(u))$. Thus, $F(u) + e(u)\, S \subset G(u) \subset G(x) + (e(u)/4)\, S$ and, by Proposition 2.2,

$$F(u) + \tfrac{3}{4} e(u)\, S \subset G(x) \qquad \text{for each } x \in S_u. \qquad (5.4)$$

Consequently, putting $H_u = F(u) + (e(u)/2)\, S$, we have

$$F(x) + \frac{e(u)}{4} S \subset H_u, \qquad H_u + \frac{e(u)}{4} S \subset G(x) \qquad \text{for each } x \in S_u.$$

The family $\mathscr{S} = \{S_u\}_{u \in X}$ is an open covering of X and so there is a partition $\mathscr{Q} = \{q_u\}_{u \in X}$ of unity subordinated to $\mathscr{S}$. Now, set

$$H(x) = \overline{H^*(x)} \qquad \text{where} \qquad H^*(x) = \sum_{u \in X} q_u(x)\, H_u, \qquad x \in X \quad (5.5)$$

$$r(x) = \tfrac{1}{4} \sum_{u \in X} q_u(x)\, e(u), \qquad x \in X. \quad (5.6)$$

By a standard argument one can easily see that (5.5) and (5.6) define, respectively, a continuous multifunction $H\colon X \to \mathscr{U}_0(Z)$ and a strictly positive continuous function $r\colon X \to \mathbb{R}$. Let us check that, with such choice of H and r, (5.1) and (5.2) are satisfied.

Indeed, let $x_0 \in X$ and, as usual, denote by $\{q_{u_i}\}_{i=1}^{s}$ the (nonempty) finite set of all functions $q_u \in \mathscr{Q}$ the supports of which contain x_0. Evidently, we have

$$H^*(x_0) = \sum_{i=1}^{s} q_{u_i}(x_0)\, H_{u_i}, \qquad r(x_0) = \tfrac{1}{4} \sum_{i=1}^{s} q_{u_i}(x_0)\, e(u_i).$$

For each $i = 1, 2, \ldots, s$ we have $x_0 \in \operatorname{supp} q_i \subset S_{u_i}$ and so, by the first relation in (5.3), $F(x_0) \subset F(u_i) + (e(u_i)/4)\, S$. Hence,

$$\begin{aligned} F(x_0) + r(x_0)\, S &\subset \sum_{i=1}^{s} q_{u_i}(x_0) \left[F(u_i) + \frac{e(u_i)}{4} S \right] \\ &\quad + \sum_{i=1}^{s} q_{u_i}(x_0) \frac{e(u_i)}{4} S \\ &\subset \sum_{i=1}^{s} q_{u_i}(x_0) \left[F(u_i) + \frac{e(u_i)}{2} S \right] \\ &\subset \sum_{i=1}^{s} q_{u_i}(x_0)\, H_{u_i} = H^*(x_0) \subset H(x_0), \end{aligned}$$

and so, since $x_0 \in X$ is arbitrary, (5.1) is fulfilled. Next, let us prove (5.2). We have

$$H^*(x_0) + r(x_0)\, S = \sum_{i=1}^{s} q_{u_i}(x_0)\left[F(u_i) + \frac{e(u_i)}{2}\, S\right] + \sum_{i=1}^{s} q_{u_i}(x_0)\,\frac{e(u_i)}{4}\, S = \sum_{i=1}^{s} q_{u_i}(x_0)\left[F(u_i) + \frac{3}{4}\, e(u_i)\, S\right].$$

Evidently, $x_0 \in \operatorname{supp} q_i \subset S_{u_i}$ for each $i = 1, 2, \ldots, s$ thus, by virtue of (5.4), $F(u_i) + \frac{3}{4} e(u_i)\, S \subset G(x_0)$. Using this in the above equality, we obtain

$$H^*(x_0) + r(x_0)\, S \subset \sum_{i=1}^{s} q_{u_i}(x_0)\, G(x_0) = G(x_0).$$

This implies that $H(x_0) + r(x_0)\, S \subset G(x_0)$ and so, since $x_0 \in X$ is arbitrary, also (5.2) is true. The last statement is obvious. This completes the proof.

REFERENCES

1. J. DIEUDONNÉ, Une généralisation des espaces compacts, *J. Math. Pures Appl.* **23** (9) (1944), 65–76.
2. J. DUGUNDJI, "Topology," Allyn & Bacon, Boston, 1966.
3. K. FAN, Fixed points and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 121–126.
4. G. HADDAD, Topological properties of the sets of solutions for functional differential inclusions, *Nonlinear Anal.* **5** (1981), 1349–1366.
5. C. J. HIMMELBERG AND F. S. VAN VLECK, A note on the solution sets of differential inclusions, *Rocky Mountain J. Math.* **12** (1982), 621–625.
6. M. HUKUHARA, Sur l'application semi-continue dont la valeur est un compact convexe, *Funkcial. Ekvac.* **10** (1967), 43–66.
7. S. KAKUTANI, A generalization of Brouwer's fixed point theorem, *Duke Math. J.* **8** (1941), 457–459.
8. J. M. LASRY ET R. ROBERT, "Analyse non lineaire multivoque," Cahiers de Mathématiques de la Décision, No. 7611, Centre de Recherche de Mathématiques de la Decision, Université de Paris IX Dauphine.
9. E. MICHAEL, Continuous selections, I, *Ann. of Math.* (2) **63** (1956), 361–382.
10. H. RÅDSTRÖM, An embedding theorem for spaces of convex sets, *Proc. Amer. Math. Soc.* **3** (1952), 165–169.
11. S. SZUFLA, On the structure of solution sets of differential and integral equations in Banach spaces, *Ann. Polon. Math.* **34** (1977), 165–177.
12. S. SZUFLA, On infinite systems of Volterra integral equations in Banach spaces, *Ann. Polon. Math.* **38** (1980), 187–194.